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BUOYANCY EFFECTS ON HORIZONTAL BOUNDARY-LAYER FLOW AND HEAT TRANSFER

E. M. SPARROW and W. J. MINKOWYCZ
Heat Transfer Laboratory, Department of Mechanical Engineering, University of Minnesota, Minneapolis, Minnesota

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Abstract—The conditions have been determined under which there are significant effects of buoyancy on a forced-convection, boundary-layer flow along a flat plate. It is found that low-Prandtl number fluids are more sensitive to buoyancy effects.

INTRODUCTION

BUOYANCY forces may exist in a boundary-layer flow over a *horizontal* surface if the surface temperature differs from that of the free stream. The effect of the buoyancy is to induce a longitudinal pressure gradient and, by this, the velocity and heat-transfer characteristics of the basic forced-convection flow may be modified. The conditions under which the buoyancy effects are significant are thus of practical interest, and it was to determine these conditions that this investigation was undertaken.

When this study was begun, there was (within the knowledge of the authors) no prior work on the problem reported in the literature. However, while calculations were in progress, a reprint appeared [1] in which the buoyancy effects were analyzed and results reported for gas flows having a Prandtl number of 0·72. The present investigation was conceived with the somewhat broader scope of studying the buoyancy effects at various Prandtl numbers, particularly because it was expected that these effects would be larger for low Prandtl fluids. The purpose of this brief paper is to present results at various Prandtl numbers and, additionally, to illuminate certain aspects of the analysis and results with a somewhat greater physical perspective. It may be noted at the outset that the effects of the buoyancy as found here are of *opposite sign* compared to those found in [1].* In other words, while we find that buoyancy increases the heat transfer and skin friction for flow above a flat plate whose surface is at a temperature higher than the free-stream flow, [1] finds an opposite effect. That the direction of the buoyancy effects is correctly represented here will be verified by physical arguments relating to the induced longitudinal pressure gradient.

* There appears to be a sign error in equation (21) of [1], wherein the term $8\eta\theta_0'$ ought to have a minus sign.

ANALYSIS

Let us consider a horizontal flat plate over which flows a laminar boundary-layer flow with free-stream velocity U_∞ and free-stream temperature T_∞. The plate temperature is T_w. The co-ordinate x measures the distance along the plate from the leading edge, while y measures the distance normal to the surface (positive, vertically upward).

In order to formulate a buoyancy force, it is necessary that density variations within the fluid be considered. In this analysis, we adopt the point of view (common in free-convection boundary-layer studies) of separating buoyancy effects from other variable property effects. In this spirit, density variations are considered only to the extent that they contribute to the buoyancy force. Aside from this, other variations in density, as well as variations in thermal conductivity and viscosity, are not included.

Considering flow above the plate, the static pressure at some location x, y within the boundary layer can be written as

$$p(x, y) - p(h) = g \int_y^h \rho \, dy \qquad (1)$$

where $p(h)$ is the pressure at some position

$y = h$ well outside the boundary layer and is, therefore, independent of x. The longitudinal pressure gradient is then found by differentiation

$$\frac{\partial p}{\partial x} = g \frac{\partial}{\partial x} \int_y^h \rho \, dy. \tag{2}$$

If the isotherms within the boundary layer were horizontal and the boundary-layer thickness uniform, then the integral would be independent of x, and $\partial p/\partial x$ would be zero. Thus, the pressure gradient is induced because the boundary layer grows thicker with x. Then, introducing a simplified equation of state

$$\rho - \rho_\infty = \beta\rho(T_\infty - T)$$

which has been widely adopted in buoyancy-affected flows (e.g. [2]), there follows

$$\frac{\partial p}{\partial x} = -g\beta\rho \frac{\partial}{\partial x} \int_y^\infty (T - T_\infty) \, dy \tag{3}$$

where the upper limit has been taken as infinity without any loss of generality. For flow below the plate, the co-ordinate y would be reversed to measure distances vertically downward, and the minus sign in equation (3) would be deleted.

It is of some interest to consider the sign of the induced pressure gradient. Consider, as a concrete example, a flow above the plate for which $T_w > T_\infty$. Then, remembering that the boundary layer thickens with increasing x, it is easy to see that $\int_y^h \rho \, dy$ decreases with x and $\int_y^\infty (T - T_\infty) \, dy$ increases with x. Thus, from either of equations (2) or (3), it follows that the induced pressure gradient $\partial p/\partial x$ is negative. Thus, the flow in the boundary layer is accelerated, thereby increasing the heat transfer and skin friction. The same conclusion applies to a flow below the plate for which $T_w < T_\infty$. The opposite effect occurs for flow above the plate for $T_w < T_\infty$ and for flow below the plate for $T_w > T_\infty$.

The relationship between the induced pressure gradient and the buoyancy force was derived in a more formal way in [1] which made physical interpretation difficult. This is perhaps why buoyancy effects of opposite sign (apparently due to a later sign error) went undetected.

With the pressure gradient given by equation (3), the mass, momentum and energy equations for boundary-layer flow above the plate become

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta \frac{\partial}{\partial x} \int_y^\infty (T - T_\infty) \, dy + \nu \frac{\partial^2 u}{\partial y^2} \tag{5}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \left(\frac{k}{\rho c_p}\right) \frac{\partial^2 T}{\partial y^2} \tag{6}$$

where the symbols have their usual meaning. In these equations, fluid-property variations have been treated as previously discussed. In addition, the viscous-dissipation and compression-work terms have not been included in the energy equation. It can be shown that both viscous dissipation and compression work are proportional to the Eckert number, $U_\infty^2/2c_p\,(T_w - T_\infty)$. Therefore, both of these energy terms will have a negligible effect on heat transfer if the Eckert number is sufficiently small;* and this is the condition for which the present analysis has been carried through. If viscous dissipation and compression work were to be considered along with the buoyancy effects, then the Eckert number would appear as an additional parameter which would have to be specified for each solution one might wish to obtain.

Because of the nature of the coupling between equations (5) and (6), it is not possible to use the usual boundary-layer mathematics (similarity transformation) by which the partial-differential equations are reduced to ordinary differential equations. Therefore, a series solution may be sought for. Inasmuch as we are seeking a solution which gives the perturbation of a basic forced convection flow due to buoyancy, it is logical to construct a series whose leading term is that for purely forced flow and the later terms give the buoyancy effect. It is, therefore, convenient to define new independent variables as follows

* The quantitative criterion is that the product of the Eckert number and the recovery factor is negligibly small compared to unity. For gases and fluids of higher Prandtl number (recovery factor $\simeq Pr^{1/2}$), the Eckert number would have to be very much less than unity. For liquid metals [recovery factor $\simeq Pr^{1/2}(0{\cdot}924 + 0{\cdot}194\,Pr^{1/2})$], [3], it would be necessary only that the Eckert number be moderately less than unity. The foregoing remarks continue to apply when there are small buoyancy effects, as considered here.

$$\eta = \frac{y}{x} Re_x^{1/2}, \qquad \xi = Gr_x/Re_x^{5/2} \qquad (7a)$$

where

$$Re_x = U_\infty x/\nu, \quad Gr_x = g\beta|T_w - T_\infty|x^3/\nu^2 \quad (7b)$$

The η variable is seen to be the well-known forced-convection (Blasius) similarity variable; while ξ measures the strength of the free convection (Grashof number) relative to the forced convection (Reynolds number) and is proportional to $x^{1/2}$. The absolute magnitude signs in the Grashof number permit consideration of both $T_w > T_\infty$ and $T_w < T_\infty$. New dependent variables f and θ, respectively related to the velocity and temperature, are then introduced

$$f(\xi, \eta) = \psi/\sqrt{(x\nu U_\infty)} = f_0(\eta) \pm \xi f_1(\eta) + \ldots \quad (8a)$$

$$\theta(\xi, \eta) = (T - T_\infty)/(T_w - T_\infty) = \theta_0(\eta) \pm \xi\theta_1(\eta) + \ldots \quad (8b)$$

where ψ is the stream function which satisfies equation (4): $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$. The functions f_0 and θ_0 are associated with the pure forced flow. The plus-minus signs refer respectively to $T_w > T_\infty$ and $T_w < T_\infty$ for flow above the plate, and have opposite meaning for flow below the plate.

Ordinary differential equations for the f and θ functions are found by substituting the series into the momentum and energy conservation equations (4) and (5) and then grouping terms according to the powers of ξ which multiply them. Boundary conditions are derived by noting that $u = v = 0$ and $T = T_w$ at $y = 0$ (plate surface) and that $u \to U_\infty$ and $T \to T_\infty$ as $\eta \to \infty$ (free stream).

SOLUTIONS

The ordinary differential equations, thus derived, were solved numerically by the Kutta–Runge integration technique on a Remington Rand 1103 electronic digital computer. The calculations were carried out for Prandtl numbers of 0·01, 0·7 and 10. From these solutions we may obtain derivatives of the θ and f functions at the plate surface ($\eta = 0$) which are required in the heat-transfer and skin-friction calculations. These have been listed in Table 1, where the primes denote differentiation with respect to η. For $Pr = 0·7$, our results differ in magnitude by 2 or 3 in the third place from those obtained in [1]* for $Pr = 0·72$. This difference may in part be due to the fact that the integration step size of [1] appears to be 8 times that of the present investigation. There is a difference in sign which has already been noted.

HEAT TRANSFER AND SKIN FRICTION

The local heat transfer from the wall to the fluid may be calculated from Fourier's Law: $q = -k(\partial T/\partial y)_{y=0}$. When the dimensionless variables of equations (7) and (8) are introduced, the expression for q becomes

$$\frac{q}{q_0} = 1 \pm \frac{Gr_x}{Re_x^{5/2}} \frac{\theta_1'(0)}{\theta_0'(0)} + \ldots \qquad (9)$$

where q_0 represents the heat-transfer result for pure forced convection flow as given by

$$q_0 x/k(T_w - T_\infty) = Re_x^{1/2}\,[-\theta_0'(0)]. \qquad (9a)$$

The departure of q/q_0 from unity is a measure of the effect of buoyancy on the local heat transfer. For flow above the plate, $q/q_0 > 1$ when $T_w > T_\infty$; while $q/q_0 < 1$ when $T_w < T_\infty$. Opposite effects are found for flow beneath the plate. The above relationships are of opposite sign to those stated in [1].

* The $f_1''(0)$ and $\theta_1'(0)$ must be respectively scaled by factors of $\frac{1}{4}$ and $\frac{1}{2}$ in order to make the comparison.

Table 1. Table of derivatives at $\eta = 0$

Pr	f_0''	$-\theta_0'$	f_1''	$-\theta_1'$	f_1''/f_0''	θ_1'/θ_0'	$\frac{1}{2}\int_0^\infty \theta_0\, d\eta$
0·01	0·33206	0·0516	15·28	0·303	46·02	5·872	5·958
0·7	0·33206	0·2927	1·722	0·357	5·186	1·220	0·9791
10·0	0·33206	0·7282	0·3851	0·229	1·160	0·3145	0·3891

The local skin friction is obtained by applying the Newtonian shear formula: $\tau = \mu(\partial u/\partial y)_{y=0}$. After introducing the dimensionless variables of equations (7) and (8), this becomes

$$\frac{\tau}{\tau_0} = 1 + \frac{Gr_x}{Re_x^{5/2}} \frac{f_1''(0)}{f_0''(0)} + \ldots \tag{10}$$

where τ_0 denotes the wall shear stress for forced convection in the absence of buoyancy and is given by

$$\frac{\tau_0}{\frac{1}{2}\rho U_\infty^2} Re_x^{1/2} = 2f_0''(0). \tag{10a}$$

The departure of τ/τ_0 from unity is a measure of the buoyancy effect. Whether $\tau/\tau_0 > 1$ or < 1 depends upon the relative size of T_w and T_∞ in exactly the same way as discussed above for the heat transfer.

By inspection of equations (9) and (10), it is seen that the magnitude of the buoyancy effect depends on $Gr_x/Re_x^{5/2}$ and on the ratios $\theta_1'(0)/\theta_0'(0)$ (heat transfer) and $f_1''(0)/f_0''(0)$ (shear stress). Taking cognizance of the definitions of the Reynolds and Grashof numbers, equation (7b), it follows that buoyancy effects will be most important for low velocity flows with large temperature differences between the surface and the free stream. Additionally, since $Gr_x/Re_x^{5/2} \sim x^{1/2}$, it follows that buoyancy effects will grow in importance with increasing distance from the leading edge.

The ratios $\theta_1'(0)/\theta_0'(0)$ and $f_1''(0)/f_0''(0)$, which are listed in Table 1, hold the key to the magnitude of the buoyancy effect. Inspection of the table in conjunction with equations (9) and (10) reveals that the skin friction is more strongly effected by buoyancy than is the heat transfer. It is also seen that the buoyancy effects on both heat transfer and skin friction are strong functions of Prandtl number, with the buoyancy growing more important as the Prandtl number decreases. For the skin friction, the buoyancy effect is forty-fold as strong for a Prandtl number of 0·01 as for a Prandtl number of 10. For the heat transfer, the corresponding increase in buoyancy strength is eighteen-fold. The Prandtl number effect is thus somewhat more strongly in evidence for the skin friction than for the heat transfer.

The greater strength of the buoyancy for low-Prandtl-number fluids is related to the thicker thermal boundary layers possessed by these fluids. That the shear stress is more effected than the heat transfer is due to the fact that the *direct* effect of the buoyancy is an additional force in the momentum balance.

To illustrate the results, suppose that we consider a case when $Re_x = 10^4$ and $Gr_x = 10^8$ for flow over a plate where $T_w > T_\infty$. Then, for $Pr = 0·7$, $q/q_0 = 1·012$ and $\tau/\tau_0 = 1·052$; while for $Pr = 0·01$, $q/q_0 = 1·059$ and $\tau/\tau_0 = 1·46$. Thus, the large influence of Prandtl number is clearly demonstrated.

One of the important utilities of the analysis is to establish quantitative conditions under which the buoyancy effects are large enough to merit consideration. Suppose that a 5 per cent change in local heat transfer or local shear due to buoyancy is taken as the threshold of significant effects. Then, for the local heat transfer, it follows from equation (9) that significant buoyancy effects are found when

$$Gr_x \geq 0·05\,[\theta_0'(0)/\theta_1'(0)]\,Re_x^{5/2} \tag{11a}$$

while, for the local shear, the corresponding criterion is

$$Gr_x \geq 0·05\,[f_0''(0)/f_1''(0)]\,Re_x^{5/2}. \tag{11b}$$

These relations have been respectively plotted on logarithmic co-ordinates on Figs. 1 and 2. To illuminate the presentation, consider the $Pr = 0·01$ curve on Fig. 1. This curve divides the Gr–Re plane into two parts. Gr–Re combinations which fall above the curve lie in the region of significant buoyancy effects; while Gr–Re combinations which fall below the curve lie in the region of negligible buoyancy effects. Each of the curves serves a similar function for each Prandtl number. For low Reynolds numbers, significant buoyancy effects occur with moderate Grashof numbers, while for high Reynolds numbers, large Grashof numbers are required. The greater sensitivity of low-Prandtl-number fluids to buoyancy is clearly demonstrated.

Thus far, discussion has been centered about the local heat transfer and shear stress. We may also consider the total heat flux and total drag as defined by

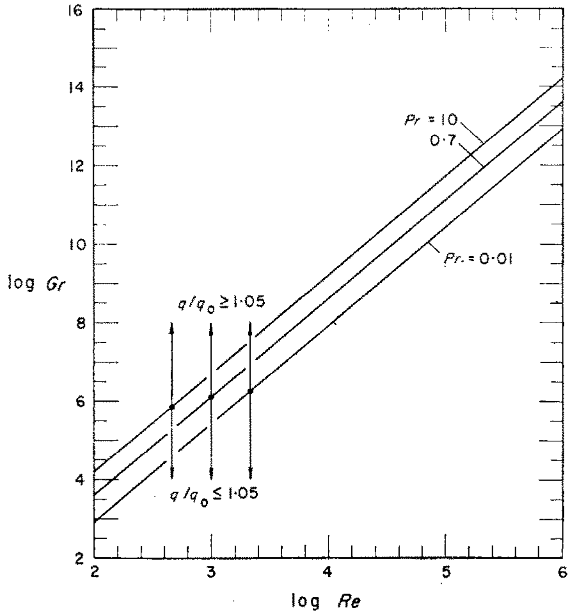

FIG. 1. Grashof–Reynolds relation for 5 per cent buoyancy effect on local heat transfer.

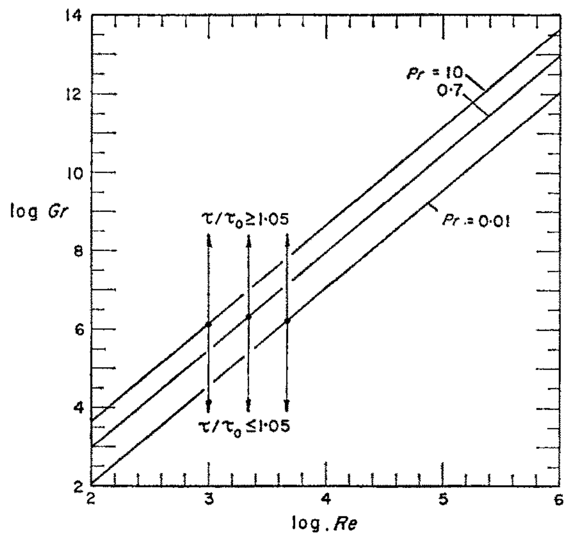

FIG. 2. Grashof–Reynolds relation for 5 per cent buoyancy effect on local shear stress.

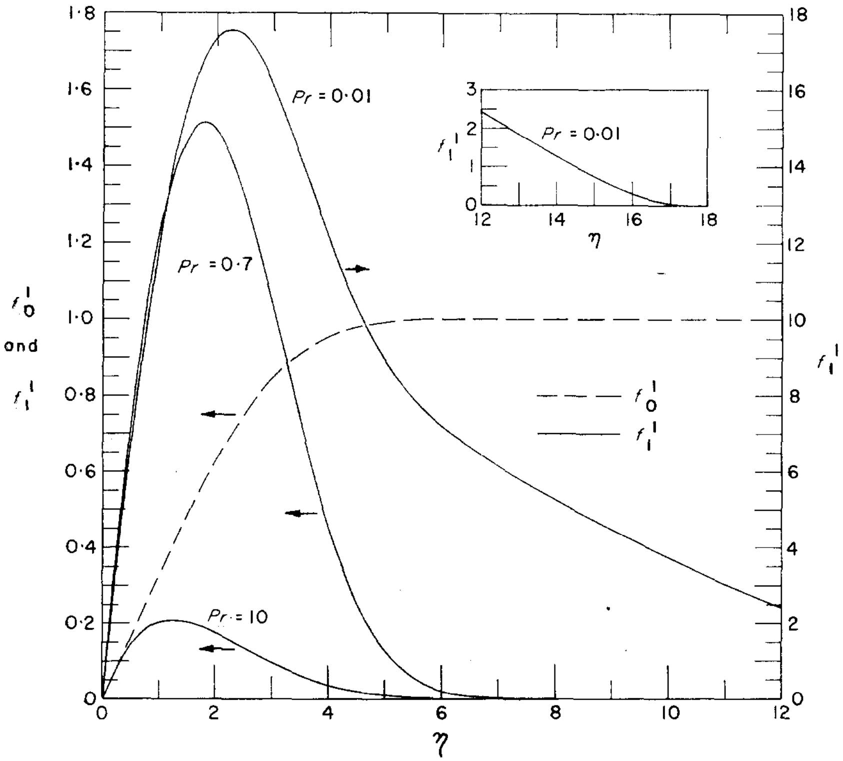

FIG. 3. The velocity functions f_0' and f_1'.

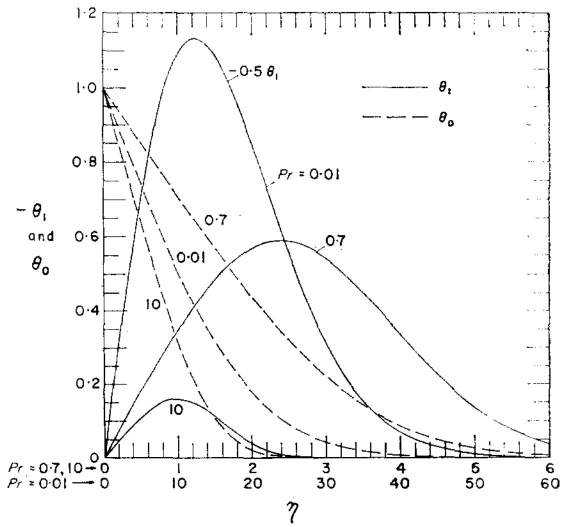

FIG. 4. The temperature functions θ_0 and θ_1.

$$Q = \int_0^x q \, dx, \qquad D = \int_0^x \tau \, dx \qquad (12)$$

For these, the buoyancy effects may be represented as Q/Q_0 and D/D_0. The resulting expressions are respectively identical to equations (9) and (10), except for a factor of 0·5 multiplying the second term of the series. Thus, the over-all quantities Q and D are less sensitive to buoyancy than are the local quantities q and τ.

VELOCITY AND TEMPERATURE PROFILES

The velocity distribution may be expressed in terms of the variables of the analysis by writing that $u = \partial\psi/\partial y$ and then introducing equations (7) and (8). Thus,

$$u/U_\infty = f_0'(\eta) \pm [Gr_x/Re_x^{5/2}] f_1'(\eta) + \ldots \qquad (13)$$

where $f_0'(\eta)$ is the well-known Blasius function for forced convection which is independent of Prandtl number. The plus sign applies to flow above the plate with $T_w > T_\infty$ and to flow below the plate with $T_w < T_\infty$. The minus sign applies in the opposite situations. The functions f_0' and f_1' are plotted on Fig. 3, where it is to be noted that the f_1' curve for $Pr = 0·01$ has its own scale on the right-hand ordinate. The velocity profiles are obtained by addition or subtraction after the $f_1'(\eta)$ curve has been multiplied by $Gr_x/Re_x^{5/2}$. Inspection of the figure reveals that the velocity profile in low-Prandtl-number fluids is much more effected by buoyancy than in high-Prandtl-number fluids. For those conditions where the contributions of $f_1'(\eta)$ are additive, the effects of buoyancy can cause u/U_∞ to exceed unity in some parts of the boundary layer.

The boundary-layer temperature profile is expressed in terms of the variables of the analysis by equation (8b). The plus and minus signs correspond to the same conditions as were discussed with relation to equation (13). Curves representing the $\theta_0(\eta)$ and $\theta_1(\eta)$ functions for the various Prandtl numbers have been plotted and are presented in Fig. 4. The temperature distributions may be obtained by addition (or subtraction) of the θ_0 and θ_1 curves, after the latter has been multiplied by $Gr_x/Re_x^{5/2}$. Physical reasoning suggests that $0 \leqslant \theta \leqslant 1$, and this highlights the fact that only moderate values of $Gr_x/Re_x^{5/2}$ may be used to be consistent with the truncation of the series.

REFERENCES

1. Y. MORI, Buoyancy effects in forced convection flow over a horizontal flat plate. *A.S.M.E.* Reprint 60–WA–220; *J. Heat Transfer*, C**83**, 479–482 (1961).
2. E. R. G. ECKERT and R. M. DRAKE, *Heat and Mass Transfer*, p. 327. McGraw-Hill, New York (1959).
3. E. M. SPARROW and J. L. GREGG, Viscous dissipation in low Prandtl number boundary-layer flow, *J. Aero. Space Sci.* **25**, 717–718 (1958).

Résumé—L'auteur détermine les conditions dans lesquelles la convection naturelle à des effets notables sur l'écoulement de la couche limite sur une plaque plane, en convection forcée. Il montre que les fluides à bas nombre de Prandtl sont plus sensibles aux effets de la convection naturelle.

Zusammenfassung—Der Einfluss des Auftriebs auf eine Grenzschichtströmung längs einer ebenen Platte bei Zwangskonvektion wurde untersucht. Flüssigkeiten mit kleiner Prandtl-Zahl sind gegen Auftriebseinflüsse besonders empfindlich.

Аннотация—Определены условия, в которых подъёмная сила жидкости оказывает значительное влияние на вынужденную конвекцию в пограничном слое продольно обтекаемой плоской пластины. Найдено, что жидкости с малым числом Прандтля являются наиболее чувствительными к эффектам подъёмной силы.